4568. *Proposed by Song Qing, Leonard Giugiuc and Michael Rozenberg.*

Let k be a fixed positive real number. Consider positive real numbers x, y and z such that

$$xy + yz + zx = 1 \quad \text{and} \quad (1 + y^2)(1 + z^2) = k^2(1 + x^2).$$

Express the maximum value of the product xyz as a function of k.

We received 9 submissions of which 6 were correct and complete. We present the solution by Arkady Alt, slightly modified.

Since $xy + yz + zx = 1$, the equation $(1 + y^2)(1 + z^2) = k^2(1 + x^2)$ is equivalent to each of:

$$\begin{aligned}
&\left(xy + yz + zx + y^2\right)\left(xy + yz + zx + z^2\right) = k^2\left(xy + yz + zx + x^2\right),\\
&(y + z)(x + y)(y + z)(x + z) = k^2(x + z)(x + y),\\
&(y + z)^2 = k^2,\\
&y + z = k.
\end{aligned}$$

Let $t = xyz$, then since $y + z = k$, we have

$$1 = (xy + zx) + yz = kx + \frac{t}{x}$$

and so $t = x(1 - kx)$. Thus, $yz = 1 - kx$ and $y + z = k$, so, by the AM-GM inequality,

$$1 - kx = yz \leqslant \frac{(y + z)^2}{4} = \frac{k^2}{4}.$$

Hence, $x \geqslant \frac{1}{k} - \frac{k}{4}$ and we are to maximize $h(x) = x(1 - kx)$ when $x \geqslant \frac{1}{k} - \frac{k}{4}$.

Since $h'(x) = 1 - 2kx$, $h(x)$ is decreasing when $\dfrac{1}{2k} < \dfrac{1}{k} - \dfrac{k}{4}$. That is, when $0 < k < \sqrt{2}$. For such k,

$$\max t = h\left(\frac{1}{k} - \frac{k}{4}\right) = \left(\frac{1}{k} - \frac{k}{4}\right)\left(1 - k\left(\frac{1}{k} - \frac{k}{4}\right)\right) = \frac{k\left(4 - k^2\right)}{16}.$$

Likewise, if $k \geqslant \sqrt{2}$, then $\dfrac{1}{k} - \dfrac{k}{4} \leqslant \dfrac{1}{2k}$ so $\frac{1}{2k}$ is in the domain of $h(x)$ and

$$\max t = h\left(\frac{1}{2k}\right) = \frac{1}{2k}\left(1 - k \cdot \frac{1}{2k}\right) = \frac{1}{4k}.$$

Thus, $\max(xyz) = \begin{cases} \dfrac{k\left(4 - k^2\right)}{16} \text{ if } k \in \left(0, \sqrt{2}\right) \\ \dfrac{1}{4k} \text{ if } k \geq \sqrt{2} \end{cases}$